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On Norms of Lewanowicz Operators

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1. INTRODUCTION

As usual, let $C[-1, 1]$ be the space of all continuous, real-valued functions on $[-1, 1]$ with the supremum norm, Π_n the subspace of all polynomials of degree less than or equal to n, and $\mathscr{P}_n$ the set of all linear projections P: $C[-1, 1] \to \Pi_n$. If $E_n[f]$ denotes the distance from f to its proximum in Π_n, then the Lebesgue inequality states that

$$\|f - Pf\|_\infty \leqslant (1 + \|P\|)\, E_n[f].$$

It is therefore sensible to find projections whose norms are small.

If $(P_n)_{n \in \mathbb{N}}$ is a sequence of projections in $\mathscr{P}_n$, the hitherto best known asymptotic equality (cf. [3]) is

$$\|P_n\| = \frac{4}{\pi^2} \ln n + O(1),$$

which, for example, holds for the Chebyshev partial sum operators, S_n, defined by the equalities

$$S_n[f] := \sum_{k=0}^{n}{}' a_k[f]\, T_k, \qquad a_k[f] = \frac{2}{\pi} \int_0^\pi f(\cos t) \cos kt\, dt,$$

$$T_k(x) = \cos(k \arccos x).$$

Here $\sum'$ indicates that the first summand should be halved.

Based on a numerical investigation of certain operators, Lewanowicz [1] conjectured that it is possible to reduce the constant coefficient of $\ln n$. That this is impossible with Lewanowicz operators will be shown in Section 3 of this paper. In Section 4 it will also be shown that there are sequences of projections whose elements require relatively few function values and whose norms have coefficients of $\ln n$ arbitrarily close to the value $4/\pi^2$.

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Since the norms of the Lewanowicz operators are smaller than those of the Chebyshev partial sum operators when n is small, these operators ought to be of some practical value. By modifying the Lewanowicz operators slightly, in Section 5 it will be indicated that many of the hitherto smallest norms can be reduced.

2. The Lewanowicz Operators and Lebesgue Functions

In [1] Lewanowicz introduced the operators,

$$I_n^{(m)}[f] = \sum_{k=0}^{n}{}' \alpha_k^{(m)}[f]\, T_k; \qquad m \geqslant n,$$

where

$$\alpha_k^{(m)}[f] = \frac{2}{m+1} \sum_{j=1}^{m+1} f(t_{m+1,j})\, T_k(t_{m+1,j}), \qquad t_{m+1,j} = \cos\theta_j,$$

$$\theta_j = \theta_{m+1,j} = \frac{2j-1}{2m+2}\pi.$$

In order to estimate the norms of the projections,

$$P_n[f] = \sum_{j=1}^{m} f(x_{n,j})\, p_{n,j}; \qquad p_{n,j} \in \Pi_n,$$

the Lebesgue functions

$$\Lambda_{P_n} = \sum_{j=1}^{m} |p_{n,j}|,$$

are used, for which the well-known equality,

$$\|P_n\| = \|\Lambda_{P_n}\|_\infty,$$

holds. (Lewanowicz assumed that, when $P_n = I_n^{([3n/2])}$, then $\|P_n\| = \Lambda_{P_n}(1)$ $(\approx (2/\sqrt{3}\pi)\ln n < (4/\pi^2)\ln n)$, which shall be disproved in the following section.)

As in [1], the following equality holds for Lewanowicz operators,

$$\Lambda_{I_n^{(m)}}(\cos t) = \frac{1}{m+1} \sum_{j=1}^{m+1} |D_n(\theta_j - t) + D_n(\theta_j + t)|,$$

where

$$D_n(u) = \sum_{k=0}^{n}{}' \cos ku = \frac{1}{2}\sin\left(n+\frac{1}{2}\right)u \cdot \csc\frac{u}{2}.$$

3. A Lower Bound for the Norms

Theorem 1. *There is a constant C such that, for every n and m with $m \geqslant n$,*

$$\|I_n^{(m)}\| \geqslant \frac{4}{\pi^2} \ln n + C.$$

Proof. Using the inequality,

$$\int_0^\pi |D_n(\theta+t)| \sin t \, dt \leqslant \int_0^\pi \left| \sin\left(n+\frac{1}{2}\right)(\theta+t)\right| dt < \pi,$$

for every $\theta \in [0, \pi]$, it follows that

$$\begin{aligned}
\|I_n^{(m)}\| &\geqslant \frac{1}{2} \int_{-1}^{1} \Lambda_{I_n^{(m)}}(x)\, dx \\
&= \frac{1}{2(m+1)} \int_0^\pi \sum_{j=1}^{m+1} |D_n(\theta_j - t) + D_n(\theta_j + t)| \sin t \, dt \\
&\geqslant \frac{1}{2(m+1)} \sum_{j=1}^{m+1} \int_0^\pi (|D_n(\theta_j - t)| - |D_n(\theta_j + t)|) \sin t \, dt \\
&> \frac{1}{m+1} \sum_{j=1}^{[(m+1)/2]} \int_0^\pi |D_n(\theta_j - t)| \sin t \, dt - \frac{\pi}{2}.
\end{aligned}$$

Furthermore, for every $\theta \in [\pi/(2n+1), \pi/2]$,

$$\begin{aligned}
&\int_0^\pi |D_n(\theta - t)| \sin t \, dt \\
&= \frac{1}{2} \int_{-\theta}^{\pi-\theta} \left| \sin\left(n+\frac{1}{2}\right) t \csc \frac{1}{2} t \right| \sin(\theta+t) \, dt \\
&\geqslant \frac{1}{2} \int_{-\theta}^{\theta} \left| \sin\left(n+\frac{1}{2}\right) t \csc \frac{1}{2} t \right| \left(\sin\theta + 2\cos\left(\theta + \frac{t}{2}\right) \sin \frac{t}{2} \right) dt \\
&\geqslant \sin\theta \int_0^\theta \left| \sin\left(n+\frac{1}{2}\right) t \csc\frac{1}{2} t \right| dt - 2\theta \\
&\geqslant \sin\theta \sum_{\nu=1}^{[(2n+1)\theta/2\pi]} \csc \frac{\nu\pi}{2n+1} \int_0^{2\pi/(2n+1)} \left| \sin\left(n+\frac{1}{2}\right) t \right| dt - \pi \\
&\geqslant \frac{4}{\pi} \sin\theta \int_{\pi/(2n+1)}^{\theta/2} \csc x \, dx - \pi \\
&\geqslant \frac{4}{\pi} \sin\theta \ln n - \frac{24}{e\pi\sqrt{3}} - \pi,
\end{aligned}$$

where the last inequality results from an elementary computation. It therefore follows that there is a constant C_1, which is independent of both n and m, for which the following inequality holds:

$$\|I_n^{(m)}\| \geqslant \frac{4 \ln n}{\pi(m+1)} \sum_{j=1}^{[(m+1)/2]} \sin \theta_j + C_1 \geqslant \frac{4}{\pi^2} \ln n + C_1 - \frac{1}{2}. \quad \text{Q.E.D.}$$

Analogously, the same statement can also be proved for the other operators introduced by Lewanowicz in [1].

4. Asymptotic Behaviour of Certain Norm Sequences

Theorem 2. *Let α and β be relatively prime natural numbers with $\alpha > \beta$, and let $m := m_n := (\alpha/\beta)\, n + \gamma_n$ and $\gamma_n = O(1)$. Then*

$$\|I_n^{(m)}\| = \frac{\pi}{2\alpha} \csc \frac{\pi}{2\alpha} \cdot \frac{4}{\pi^2} \ln n + O(1).$$

Proof. Instead of the complicated Lebesgue function, consider the symmetric $\pi/(m+1)$-periodic function

$$\psi_{n,m}(t) = \frac{1}{m+1} \sum_{j=1}^{m+1} (|D_n(\theta_j - t)| + |D_n(\theta_j + t)|),$$

so that, for every $\varepsilon \in (0, \pi/2)$ one has

$$\max_{\varepsilon \leqslant t \leqslant \pi - \varepsilon} |D_n(\theta_j + t)| \leqslant \frac{1}{2} \csc \frac{\varepsilon}{2} \leqslant \frac{\pi}{2\varepsilon},$$

and hence also

$$\|I_n^{(m)}\| = \max_{\varepsilon \leqslant t \leqslant \pi - \varepsilon} \psi_{n,m}(t) + O(1) = \max_{0 \leqslant t \leqslant \pi/(2m+2)} \psi_{n,m}(t) + O(1).$$

Moreover the summands corresponding to $j = 1$ and $j = m+1$ can be dropped without affecting the asymptotic behaviour, so that

$$\|I_n^{(m)}\| = \max_{0 \leqslant t \leqslant \pi/(2m+2)} \tilde{\psi}_{n,m}(t) + O(1),$$

where

$$\tilde{\psi}_{n,m}(t) = \frac{1}{m+1} \sum_{j=2}^{m} (|D_n(\theta_j - t)| + |D_n(\theta_j + t)|).$$

In contrast to $\psi_{n,m}$, the function $\tilde{\psi}_{n,m}$ has the advantage that every term of the form $\csc(\theta_j \pm t)$ is asymptotically $O(j^{-1}n)$.

In the sequel, $\tilde{\psi}_{n,m}$ shall be considered as a function defined on $[0, \pi/(2m+2)]$. The indices of the θ_j will now be partitioned into equivalence classes modulo α, so that

$$\theta_j = \frac{2j-1}{2m+2}\pi; \qquad j = \nu\alpha + \mu;\ \mu = 1, 2, ..., \alpha;\ \nu = 1, 2, ..., \left[\frac{n}{\beta}\right].$$

(The missing or additional summands, whose number is bounded by $\alpha + |\gamma_n|$, do not affect the asymptotic behaviour.) Since

$$\left(n+\frac{1}{2}\right)\frac{1}{2m+2} = \frac{\beta}{2\alpha} + O(n^{-1}),$$

it follows that

$$\left|\sin\left(n+\frac{1}{2}\right)(\theta_j \pm t)\right| = \left|\sin\left((2\mu-1)\frac{\beta\pi}{2\alpha} \pm \left(n+\frac{1}{2}\right)t\right)\right| + O\left(\frac{j}{n}\right),$$

and hence

$$\tilde{\psi}_{n,m}(t) = \frac{1}{2m+2}\sum_{\mu=0}^{\alpha-1}\left\{\left[\left|\sin\left((2\mu+1)\frac{\beta\pi}{2\alpha} + \left(n+\frac{1}{2}\right)t\right)\right| + \left|\sin\left((2\mu+1)\frac{\beta\pi}{2\alpha} - \left(n+\frac{1}{2}\right)t\right)\right|\right] \cdot \sum_{\nu=1}^{[n/\beta]} \csc\frac{1}{2}\theta_{\alpha\nu+\mu}\right\} + O(1).$$

Moreover,

$$\sum_{\nu=1}^{[n/\beta]} \csc\frac{1}{2}\theta_{\alpha\nu+\mu} = \frac{1}{\alpha}\sum_{j=1}^{m+1}\csc\frac{1}{2}\theta_j + O(n) = \frac{2m+2}{\alpha\pi}\ln n + O(n),$$

whence

$$\|I_n^{(m)}\| = \frac{\ln n}{\alpha\pi}\cdot\|S\|_\infty + O(1),$$

where

$$S(z) = \sum_{\mu=0}^{\alpha-1}\left\{\left|\sin\left((2\mu+1)\frac{\beta\pi}{2\alpha} + z\right)\right| + \left|\sin\left((2\mu+1)\frac{\beta\pi}{2\alpha} - z\right)\right|\right\}.$$

Since β and α are relatively prime, the sequence $\mu\beta \pmod{\alpha}$ runs through all the natural numbers from 0 to $\alpha - 1$, so that $(2\mu+1)\beta \pmod{2\alpha}$ takes each value of $\beta+2\nu \pmod{2\alpha}$ exactly once. If α is even (resp. odd), then β is odd (resp. even), and $(2\mu+1)\beta \pmod{2\alpha}$ runs through all the odd (resp. even) numbers between 0 and $2\alpha-1$, so that

$$S(z)=\begin{cases}\sum_{\mu=0}^{\alpha-1}\left\{\left|\sin\left(\frac{2\mu+1}{2\alpha}\pi+z\right)\right|+\left|\sin\left(\frac{2\mu+1}{2\alpha}\pi-z\right)\right|\right\}, & \text{if } \alpha \text{ is even;}\\ \sum_{\mu=0}^{\alpha-1}\left\{\left|\sin\left(\frac{\mu\pi}{\alpha}+z\right)\right|+\left|\sin\left(\frac{\mu\pi}{\alpha}-z\right)\right|\right\}, & \text{otherwise.}\end{cases}$$

In particular, S is a symmetric, (π/α)-periodic function. The signs of the sine terms can now easily be found whenever $z\in[0, \pi/2\alpha]$, whence S can be given explicitly by

$$S(z)=\begin{cases}2\csc(\pi/2\alpha)\cos z, & \text{whenever } \alpha \text{ is even;}\\ 2\csc(\pi/2\alpha)\cos(\pi/2\alpha-z), & \text{otherwise.}\end{cases}\qquad \text{Q.E.D.}$$

Again, the same statement remains true for the other class of operators introduced in [1]. For applications, these results imply that one can choose $m:=[(1001/1000)n]$ instead of perhaps $[(1001/2)n]$, thereby drastically reducing the number of function values without necessarily obtaining significantly worse results.

5. A Modification of the Operators $I_n^{(m)}$

If one considers the graphs of some of the Lebesgue functions of $I_n^{(m)}$, for small n and m, one observes that the maxima occur on the boundary of the basic interval. As has already been proved useful for interpolation operators (cf. [2]), by linearly stretching using the transformation K_m below, the boundary maxima can be pushed outside the basic interval. One defines

$$\tilde{I}_n^{(m)}=K_m\circ I_n^{(m)}\circ K_m^{-1},\qquad \text{where } K_m[f](x)=f(t_{m+1,1}\cdot x).$$

Since K_m and its inverse function transform polynomials into polynomials of the same degree, it is clear that $\tilde{I}_n^{(m)}$ is indeed a projection onto Π_n, having norms

$$\|\tilde{I}_n^{(m)}\|=\max_{|x|\leqslant t_{m+1,1}} \Lambda_{I_n^{(m)}}(x)\leqslant \|I_n^{(m)}\|.$$

As above, this stretching does not affect the asymptotic behaviour of the norms.

TABLE I

n	m	i_n	l_n	n	m	i_n	l_n
1	1	1.000	1.333	21	59	2.261	2.297
2	2	1.250	1.494	22	62	2.279	2.317
3	3	1.430	1.601	23	65	2.296	2.336
4	4	1.570	1.681	24	68	2.312	2.355
5	5	1.685	1.745	25	71	2.328	2.373
6	6	1.783	1.798	26	74	2.342	2.389
7	9	1.854	1.843	27	77	2.356	2.406
8	20	1.905	1.893	28	80	2.370	2.422
9	11	1.947	1.939	29	83	2.382	2.437
10	25	1.984	1.984	30	86	2.393	2.452
11	26	2.019	2.022	31	89	2.406	2.466
12	30	2.053	2.059	32	92	2.418	2.480
13	34	2.084	2.092	33	95	2.431	2.494
14	33	2.107	2.124	34	98	2.444	2.506
15	41	2.136	2.153	35	101	2.455	2.520
16	44	2.159	2.181	36	103	2.466	2.531
17	40	2.178	2.206	37	106	2.475	2.544
18	45	2.206	2.231	38	109	2.485	2.555
19	52	2.225	2.254	39	112	2.496	2.567
20	47	2.241	2.276	40	115	2.505	2.578

In Table I, the values

$$i_n := \min_{n \leqslant m \leqslant 100+n} \|\tilde{T}_n^{(m)}\|$$

have been computed numerically, where the corresponding values of m and the smallest Lewanowicz norms, l_n, have also been listed.

References

1. S. Lewanowicz, Some polynomial projections with finite carrier, *J. Approx. Theory* **34** (1982), 249–263.
2. F. W. Luttmann and T. J. Rivlin, Some numerical experiments in the theory of polynomial interpolation, *IBM J. Res. Develop.* **9** (1965), 187–191.
3. M. J. D. Powell, On the maximum errors of polynomial approximations defined by interpolation and by least squares criteria, *Comput. J.* **9** (1967), 404–407.